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Infinitely many conservation laws and integrable discretizations for some lattice soliton equations

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Abstract
In this paper, by means of the Lax representations, we demonstrate the existence of infinitely many conservation laws for the general Toda-type lattice equation, the relativistic Volterra lattice equation, the Suris lattice equation and some other lattice equations. The conserved density and the associated flux are given formulaically. We also give an integrable discretization for a lattice equation with n dependent coefficients.

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1. Introduction

The study of the lattice soliton equations has received considerable attention in recent years. A lattice soliton equation possesses rich mathematical structure and a lot of integrable properties, such as the Lax representation, the Hamiltonian structure, soliton solution, infinitely many conservation laws, and so on. The existence of infinitely many conservation laws is an important indicator of integrability of the system. From a physical viewpoint, it is also very interesting to know whether there exist conservation laws for a lattice system. For a lattice equation

$$F(\dot{u}_n, \ddot{u}_n, \ldots, u_{n-1}, u_n, u_{n+1}, \ldots) = 0 \tag{1.1}$$

where $\dot{u}_n = \frac{\partial u_n}{\partial t}$, $\ddot{u}_n = \frac{\partial^2 u_n}{\partial t^2}$, if there exist functions ρ_n and J_n, such that

$$\dot{\rho}_n|_{F=0} = J_n - J_{n+1} \tag{1.2}$$

then equation (1.2) is called the conservation law of equation (1.1), where ρ_n is the conserved density and J_n is the associated flux. Suppose equation (1.1) has conservation law (1.2) and J_n is bounded for all n and vanishes at the boundaries, then $\sum_n \rho_n = c$ is an integral of motion of lattice equation (1.1). How to obtain conservation laws for a lattice system? Hereman and

Göktas proposed a computational method for scaling invariant systems [1, 2]. By means of this method, conserved densities for many nonlinear lattice systems, such as the Toda lattice, the Volterra lattice, the Suris lattices *et al*, are derived. In [3], using the method, several conservation laws for the Belov–Chaltikian lattice [4] and the Blaszak–Marciniak three-field lattice [5] are given. However, the computational method has a shortcoming. On the one hand, if we want to obtain more higher order conservation laws, the calculation is very tedious and complex. On the other hand, the method cannot provide the justification of whether there exist infinitely many conservation laws for the discrete system. In [6, 7], by means of the linear scattering equation, infinitely many conservation laws for the semi-discrete matrix NLS equation are derived. In this paper, we focus on the following lattice equations:

- The general Toda-type lattice soliton equation [8],

$$\begin{aligned} \dot{p}_n &= -\mu p_n \left(\frac{q_{n+1}(\beta p_n - \delta)}{q_n} - \frac{q_n(\beta p_{n-1} - \delta)}{q_{n-1}} \right) \\ \dot{q}_n &= q_n(\mu p_n + b) - \beta \mu q_{n+1} p_n \end{aligned} \tag{1.3}$$

  which includes many well-known lattice equations, such as the modified Toda lattice, the relativistic Toda lattice and the Suris lattices discussed in [9–15]:

$$\begin{aligned} \ddot{q}_n &= \dot{q}_n(\mathrm{e}^{q_{n+1}-q_n} - \mathrm{e}^{q_n-q_{n-1}}) \\ \ddot{q}_n &= \dot{q}_{n+1}\dot{q}_n \frac{g^2\,\mathrm{e}^{q_{n+1}-q_n}}{1+g^2\,\mathrm{e}^{q_{n+1}-q_n}} - \dot{q}_n\dot{q}_{n-1}\frac{g^2\,\mathrm{e}^{q_n-q_{n-1}}}{1+g^2\,\mathrm{e}^{q_n-q_{n-1}}} \\ \ddot{q}_n &= \dot{q}_n\dot{q}_{n+1} \frac{g^2\,\mathrm{e}^{q_{n+1}-q_n}}{1+g^2\,\mathrm{e}^{q_{n+1}-q_n}} - \dot{q}_{n-1}\dot{q}_n\frac{g^2\,\mathrm{e}^{q_n-q_{n-1}}}{1+g^2\,\mathrm{e}^{q_n-q_{n-1}}} + \delta g^2 \dot{q}_n(\mathrm{e}^{q_{n+1}-q_n} - \mathrm{e}^{q_n-q_{n-1}}) \\ \ddot{q}_n &= (1+\epsilon\dot{q}_n)(\mathrm{e}^{q_{n+1}-q_n} - \mathrm{e}^{q_n-q_{n-1}}) \\ \ddot{q}_n &= (1+\epsilon\dot{q}_n)(1+\epsilon\dot{q}_{n+1})\frac{\mathrm{e}^{q_{n+1}-q_n}}{1+\epsilon^2\,\mathrm{e}^{q_{n+1}-q_n}} - (1+\epsilon\dot{q}_{n-1})(1+\epsilon\dot{q}_n)\frac{\mathrm{e}^{q_n-q_{n-1}}}{1+\epsilon^2\,\mathrm{e}^{q_n-q_{n-1}}} \\ \ddot{q}_n &= (1+\epsilon\dot{q}_n)\left(\frac{\dot{q}_{n+1} - \mathrm{e}^{q_{n+1}-q_n}}{1+\epsilon\,\mathrm{e}^{q_{n+1}-q_n}}\mathrm{e}^{q_{n+1}-q_n} - \frac{\dot{q}_{n-1} - \mathrm{e}^{q_n-q_{n-1}}}{1+\epsilon\,\mathrm{e}^{q_n-q_{n-1}}}\mathrm{e}^{q_n-q_{n-1}} \right). \end{aligned} \tag{1.4}$$

- The relativistic Volterra lattice equation,

$$\begin{aligned} \dot{p}_n &= p_n(q_n - q_{n-1} + h(p_n q_n - p_{n-1}q_{n-1})) \\ \dot{q}_n &= q_n(p_{n+1} - p_n + h(p_{n+1}q_{n+1} - p_n q_n)) \end{aligned} \tag{1.5}$$

  which was proposed by Suris and Ragnisco in [14]. Its bilinear structure and determinant solution was obtained [16].

- Lattice equation with n dependent coefficients,

$$\begin{aligned} \dot{p}_n &= p_n \left(\frac{1+p_{n+1}}{1+p_{n+1}+\theta(n)q_{n+1}/(q_n\sqrt{p_n})} - \frac{1+p_n}{1+p_n+\theta(n-1)q_n/(q_{n-1}\sqrt{p_{n-1}})} \right) \\ \dot{q}_n &= q_n \left(\frac{1+\theta(n-1)q_n/(2q_{n-1}\sqrt{p_{n-1}})}{1+p_n+\theta(n-1)q_n/(q_{n-1}\sqrt{p_{n-1}})} \right) \end{aligned} \tag{1.6}$$

  which comes from the following lattice equation discussed in [17]:

$$\begin{aligned} \dot{p}_n &= p_n \left(\frac{1+p_{n+1}}{1+p_{n+1}+q_{n+1}s_n/q_n} - \frac{1+p_n}{1+p_n+q_n s_{n-1}/q_{n-1}} \right) \\ \dot{q}_n &= q_n \left(\frac{1+q_n s_{n-1}/(2q_{n-1})}{1+p_n+q_n s_{n-1}/q_{n-1}} \right) \\ \dot{s}_n &= \frac{s_n}{2}\left(\frac{q_{n+1}s_n/q_n}{1+p_{n+1}+q_{n+1}s_n/q_n} - \frac{q_n s_{n-1}/q_{n-1}}{1+p_n+q_n s_{n-1}/q_{n-1}} \right). \end{aligned} \tag{1.7}$$

For equation (1.7), we have a strong constraint $\left(p_n s_n^2\right)_t = 0$, which means $p_n s_n^2$ is a function of only n. Set $p_n s_n^2 = \theta^2(n)$, with $\theta(n)$ being an arbitrary function of n, equation (1.6) is given. Integrable properties of lattice equations (1.3)–(1.6), such as the Lax representations and the Hamiltonian structures, have been obtained. The purpose of this paper is to demonstrate the existence of infinitely many conservation laws for lattice equations (1.3)–(1.6) and to give the corresponding conserved density and the associated flux formulaically by means of their Lax pairs and the method proposed in [6, 7]. Another interesting topic in discrete soliton theory is integrable discretization. For a continuous soliton equation which admits continuous Lax pairs, its spatial discrete version (differential-difference equation or lattice equation) admits semi-discrete Lax pairs

$$\psi_{n+1} = U_n \psi_n \qquad \psi_{n,t} = N_n \psi_n \tag{1.8}$$

and its spatial and temporal discrete version (difference-difference equation or partial difference equation) admits discrete Lax pairs

$$\psi_{n+1} = U_n \psi_n \qquad \tilde{\psi}_n = V_n \psi_n \tag{1.9}$$

where $\tilde{\psi}_n = \psi_n(t+h)$ with h being discrete-time stepsize. Nonlinear partial difference equations have been obtained which have as limiting forms the nonlinear Schrödinger, KdV and mKdV equations [18–20]. These partial difference equations have a number of special properties [21] and are constructed by the method of inverse scattering transform. Computations showed that these partial difference equations provide excellent numerical schemes [22–24]. Integrable discretizations for lattice equations, such as the Toda, the Volterra, the relativistic Toda, the Bogoyavlensky and the relativistic Volterra lattices, have been obtained [25–29]. Here, the phrase 'integrable discretization' for a lattice equation means the obtained partial difference equation to admit discrete Lax pairs (1.9) and it is a discrete-time approximation to the original lattice equation. In [17], we obtained integrable discretization of the general Toda-type equation (1.3). As an application, the Lagrangian and Newtonian forms of integrable discretizations of the special Toda-type lattice equation (1.4) are given uniformly. What is the integrable discretization of lattice equation (1.6)? This question was discussed in [17]. However, we failed to find a proper form. Another aim of the present paper is to give an answer to the integrable discretization of lattice equation (1.6).

The paper is organized as follows. In section 2 we give the Lax pairs for lattices (1.3)–(1.6) which come from a proper discrete linear scattering problem. In section 3, it is shown that the lattice derived from the discrete linear scattering problem has infinitely many conservation laws and the corresponding conserved density and the associated flux are given formulaically. Then we obtain the corresponding results for lattice equations (1.3)–(1.6). So, their integrability is further confirmed. In the concluding section we deal with integrable discretization of lattice equation (1.6).

2. Lax pairs for lattice equations (1.3), (1.5) and (1.6)

In order to derive infinitely many conservation laws for lattice equations (1.3)–(1.6), we give their Lax pairs in this section. Considering the discrete linear scattering problem

$$\psi_{n+1} = \begin{pmatrix} \lambda p_n - \lambda^{-1} & q_n \\ r_n & \lambda s_n \end{pmatrix} \psi_n \tag{2.1}$$

$$\frac{\partial \psi_n}{\partial t} = N_n \psi_n \qquad N_n = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \tag{2.2}$$

we have the following results:

- The general Toda-type lattice equation (1.3) admits the Lax pairs (2.1) and (2.2), where

$$s_n = \delta \qquad r_n = \frac{\beta p_n - \delta}{q_n} \tag{2.3}$$

and

$$N_n = \begin{pmatrix} \frac{-\mu q_n(\beta p_{n-1}-\delta)}{q_{n-1}} + b + d + \mu\lambda^{-2} & -\lambda^{-1}\mu q_n \\ \frac{-\lambda^{-1}\mu(\beta p_{n-1}-\delta)}{q_{n-1}} & d \end{pmatrix} \tag{2.4}$$

with β , μ, δ, b and d being arbitrary constants.

- The relativistic Volterra lattice equation (1.5) has the Lax pairs (2.1) and (2.2) with (2.3) and (2.4), where

$$p_n \to \frac{p_n}{\mu} \qquad q_n \to \mathrm{e}^{D^{-1}\ln\frac{q_n}{\mu\delta}} \qquad r_n \to \left(\frac{\beta}{\mu}p_n - \delta\right)\mathrm{e}^{-D^{-1}\ln\frac{q_n}{\mu\delta}} \qquad h = \frac{-\beta}{\mu\delta} \tag{2.5}$$

and D^{-1} is the inverse operator of difference operator D defined by $D^{-1}f_n = \frac{1}{2}\left(\sum_{j\leqslant n-1} - \sum_{j\geqslant n}\right)f_j$.

- Lattice equation (1.6) admits the Lax pairs (2.1) and (2.2), where

$$r_n = \frac{-\theta(n)(p_n+1)}{q_n\sqrt{p_n}} \tag{2.6}$$

$$N_n = \frac{1}{(1+\lambda^2)\left(1 + p_n + \frac{\theta(n-1)q_n}{q_{n-1}\sqrt{p_{n-1}}}\right)}\begin{pmatrix} \lambda^2(p_n+1) & \lambda q_n \\ -\frac{\lambda\theta(n-1)(p_n+1)}{q_{n-1}\sqrt{p_{n-1}}} & \frac{(\lambda^2-1)\theta(n-1)q_n}{2q_{n-1}\sqrt{p_{n-1}}} \end{pmatrix}. \tag{2.7}$$

3. Infinitely many conservation laws of the general Toda-type lattice equation (1.3), the relativistic Volterra lattice equation (1.5) and lattice equation (1.6)

In this section, we first show that the lattice equation derived from discrete linear scattering problem (2.1)–(2.2) has infinitely many conservation laws and the corresponding conserved density and the associated flux are given formulaically. Then we give the corresponding results for lattice equations (1.3)–(1.6). For discrete isospectral problem (2.1)–(2.2), a direct calculation leads to

$$\frac{\psi_{1n+1}}{\psi_{1n}} = \lambda p_n - \lambda^{-1} + q_n\frac{\psi_{2n}}{\psi_{1n}} \tag{3.1}$$

and

$$\frac{\left(\psi_{1n+1}\psi_{1n}^{-1}\right)_t}{\psi_{1n+1}\psi_{1n}^{-1}} = A_{n+1} + B_{n+1}\Gamma_{n+1} - A_n - B_n\Gamma_n \tag{3.2}$$

where $\Gamma_n = \frac{\psi_{2n}}{\psi_{1n}}$, and

$$\Gamma_{n+1} = (r_n + \lambda s_n\Gamma_n)\left(\lambda p_n - \lambda^{-1} + q_n\Gamma_n\right)^{-1}.$$

It follows that

$$\Gamma_n = \lambda^2 p_{n-1}\Gamma_n + \lambda q_{n-1}\Gamma_{n-1}\Gamma_n - \lambda^2 s_{n-1}\Gamma_{n-1} - \lambda r_{n-1}. \tag{3.3}$$

Suppose the eigenfunctions $\psi_1(n, t, \lambda)$ and $\psi_2(n, t, \lambda)$ are two analytical functions of the arguments and expand Γ_n with respect to λ by the Laurent series

$$\Gamma_n = \sum_{j=0}^{\infty} \lambda^j w_n^{(j)} \tag{3.4}$$

and substitute equation (3.4) into equation (3.3), we have

$$\begin{aligned}
&w_n^{(1)} = -r_{n-1} \qquad w_n^{(3)} = p_{n-1} w_n^{(1)} + q_{n-1} w_{n-1}^{(1)} w_n^{(1)} - s_{n-1} w_{n-1}^{(1)} \\
&w_n^{(5)} = p_{n-1} w_n^{(3)} + q_{n-1} \left(w_{n-1}^{(3)} w_n^{(1)} + w_{n-1}^{(1)} w_n^{(3)} \right) - s_{n-1} w_{n-1}^{(3)} \\
&w_n^{(2j-1)} = p_{n-1} w_n^{(2j-3)} + q_{n-1} \left(w_{n-1}^{(2j-3)} w_n^{(1)} + w_{n-1}^{(2j-5)} w_n^{(3)} + \cdots + w_{n-1}^{(1)} w_n^{(2j-3)} \right) \\
&\qquad - s_{n-1} w_{n-1}^{(2j-3)} \qquad j = 2, 3, 4, \ldots \\
&w_n^{(2j)} = 0 \qquad j = 0, 1, 2, \ldots .
\end{aligned} \tag{3.5}$$

It follows from equation (3.2) that

$$\frac{\partial}{\partial t} \sum_{k=1}^{\infty} \frac{\Phi_n^k}{k} = H_n - H_{n+1} \tag{3.6}$$

where

$$\Phi_n = \lambda^2 p_n + \lambda q_n \Gamma_n \qquad H_n = A_n + B_n \Gamma_n. \tag{3.7}$$

Note that

$$\Phi_n = \lambda^2 \sum_{k=0}^{\infty} \lambda^{2k} v_{2k} \tag{3.8}$$

with

$$v_0 = p_n - q_n r_{n-1} \qquad v_{2k} = q_n w_n^{(2k+1)} \tag{3.9}$$

we have

$$\frac{\partial}{\partial t} \sum_{k=0}^{\infty} \lambda^{2k} \rho_n^{(2k)} = \lambda^{-2} (H_n - H_{n+1}) \tag{3.10}$$

where

$$\begin{aligned}
\rho_n^{(2k)} = v_{2k} &+ \frac{1}{2} \sum_{l_1+l_2=2k-2} v_{l_1} v_{l_2} + \frac{1}{3} \sum_{l_1+l_2+l_3=2k-4} v_{l_1} v_{l_2} v_{l_3} + \cdots \\
&+ \frac{1}{k-1} \sum_{l_1+l_2+\cdots+l_{k-1}=4} v_{l_1} v_{l_2} \cdots v_{l_{k-1}} + v_0^{k-1} v_2 + \frac{v_0^{k+1}}{k+1}.
\end{aligned} \tag{3.11}$$

Make a comparison of the powers of λ on both sides of equation (3.10), infinitely many conservation laws of the lattice equation derived from scattering problem (2.1)–(2.2) are obtained,

$$\rho_{n,t}^{(2k)} = J_n^{(2k)} - J_{n+1}^{(2k)} \qquad k = 0, 1, 2, \ldots . \tag{3.12}$$

On the other hand, from linear scattering problem (2.1)–(2.2), we have

$$\frac{\psi_{2n+1}}{\psi_{2n}} = \lambda s_n + r_n \Gamma_n \tag{3.13}$$

and

$$\frac{\left(\psi_{2n+1}\psi_{2n}^{-1}\right)_t}{\psi_{2n+1}\psi_{2n}^{-1}} = D_{n+1} + C_{n+1}\Gamma_{n+1} - D_n - C_n\Gamma_n \tag{3.14}$$

where $\Gamma_n = \frac{\psi_{1n}}{\psi_{2n}}$, and

$$\Gamma_{n+1} = \frac{\left(\lambda p_n - \lambda^{-1}\right)\Gamma_n + q_n}{r_n\Gamma_n + \lambda s_n}.$$

It follows that

$$\Gamma_n = \lambda^2 p_n\Gamma_n + \lambda q_n - \lambda r_n\Gamma_n\Gamma_{n+1} - \lambda^2 s_n\Gamma_{n+1}. \tag{3.15}$$

Set

$$\Gamma_n = \sum_{j=0}^{\infty}\lambda^j f_n^{(j)} \tag{3.16}$$

and substitute equation (3.16) into equation (3.15), we obtain that

$$\begin{aligned}
&f_n^{(1)} = q_n\\
&f_n^{(3)} = p_n f_n^{(1)} - s_n f_{n+1}^{(1)} - r_n f_n^{(1)} f_{n+1}^{(1)}\\
&f_n^{(5)} = p_n f_n^{(3)} - s_n f_{n+1}^{(3)} - r_n\left(f_n^{(1)} f_{n+1}^{(3)} + f_n^{(3)} f_{n+1}^{(1)}\right)\\
&f_n^{(2j-1)} = p_n f_n^{(2j-3)} - s_n f_{n+1}^{(2j-3)} - r_n\left(f_n^{(1)} f_{n+1}^{(2j-3)} + f_n^{(3)} f_{n+1}^{(2j-5)} + \cdots + f_n^{(2j-3)} f_{n+1}^{(1)}\right)\\
&\qquad j = 2, 3, 4, \ldots\\
&f_n^{(2j)} = 0, \qquad j = 0, 1, 2, \ldots.
\end{aligned} \tag{3.17}$$

It follows from equation (3.14) that

$$\frac{\partial}{\partial t}\left[\ln(s_n + r_n q_n) + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}\Psi_n^k}{k}\right] = G_{n+1} - G_n \tag{3.18}$$

where

$$\Psi_n = \frac{r_n(\Gamma_n - \lambda q_n)}{\lambda(s_n + r_n q_n)} \qquad G_n = C_n\Gamma_n + D_n. \tag{3.19}$$

Note that

$$\Psi_n = \lambda^2\sum_{k=0}^{\infty}\lambda^{2k} g_{2k} \tag{3.20}$$

with

$$g_{2k} = \frac{r_n f_n^{(2k+3)}}{s_n + r_n q_n} \tag{3.21}$$

we have

$$\frac{\partial}{\partial t}\left[\ln(s_n + r_n q_n) + \lambda^2\sum_{k=0}^{\infty}\lambda^{2k}\gamma_n^{(2k)}\right] = G_{n+1} - G_n \tag{3.22}$$

where

$$\begin{aligned}
\gamma_n^{(2k)} = g_{2k} &- \frac{1}{2}\sum_{l_1+l_2=2k-2} g_{l_1}g_{l_2} + \frac{1}{3}\sum_{l_1+l_2+l_3=2k-4} g_{l_1}g_{l_2}g_{l_3} + \cdots\\
&+ \frac{(-1)^{k-2}}{k-1}\sum_{l_1+l_2+\cdots+l_{k-1}=4} g_{l_1}g_{l_2}\ldots.g_{l_{k-1}} + (-1)^{k-1}g_0^{k-1}g_2 + \frac{(-1)^k g_0^{k+1}}{k+1}.
\end{aligned} \tag{3.23}$$

Make a comparison of the powers of λ on both sides of equation (3.22), we propose another infinitely many conservation laws of the lattice equation derived from linear scattering problem (2.1)–(2.2),

$$\gamma_{n,t}^{(2k)} = z_{n+1}^{(2k)} - z_n^{(2k)} \qquad k = -1, 0, 1, 2, \ldots. \tag{3.24}$$

Example 1. For the general Toda-type equation (1.3), we have

$$\begin{aligned}
w_n^{(1)} &= \frac{\delta - \beta p_{n-1}}{q_{n-1}} \qquad w_n^{(3)} = \frac{p_{n-1}(\delta - \beta p_{n-1})}{q_{n-1}} - \frac{\beta p_{n-1}(\delta - \beta p_{n-2})}{q_{n-2}} \\
w_n^{(5)} &= \left(p_{n-1} + \frac{q_{n-1}(\delta - \beta p_{n-2})}{q_{n-2}}\right)\left[\frac{p_{n-1}(\delta - \beta p_{n-1})}{q_{n-1}} - \frac{\beta p_{n-1}(\delta - \beta p_{n-2})}{q_{n-2}}\right] \\
&\quad - \beta p_{n-1}\left[\frac{p_{n-2}(\delta - \beta p_{n-2})}{q_{n-2}} - \frac{\beta p_{n-2}(\delta - \beta p_{n-3})}{q_{n-3}}\right]
\end{aligned} \tag{3.25}$$

$\ldots$

$$H_n = a_n + \lambda^{-1} b_n \Gamma_n = -\mu q_n \sum_{j=1}^{+\infty} \lambda^{2j} w_n^{(2j+1)} \tag{3.26}$$

$$\begin{aligned}
f_n^{(1)} &= q_n \qquad f_n^{(3)} = p_n(q_n - \beta q_{n+1}) \\
f_n^{(5)} &= \beta p_n p_{n+1}(\beta q_{n+2} - q_{n+1}) + (q_n - \beta q_{n+1})\left[p_n^2 + \frac{p_n q_{n+1}(\delta - \beta p_n)}{q_n}\right]
\end{aligned} \tag{3.27}$$

$\ldots$

$$G_n = d + \frac{\mu(\delta - \beta p_{n-1})}{q_{n-1}} \sum_{j=1}^{+\infty} \lambda^{2j} f_n^{(2j+1)}. \tag{3.28}$$

So infinitely many conservation laws (3.12) and (3.24) of the general Toda-type lattice equation (1.3) are given, where the corresponding conserved densities $\rho_n^{(2k)}, \gamma_n^{(2k)}$ $(k = 0, 1, 2, 3, \ldots)$ and the associated flux $J_n^{(2k)}, z_n^{(2k)}$ $(k = 0, 1, 2, 3, \ldots)$ are presented in the following equations:

$$\begin{aligned}
\rho_n^{(0)} &= v_0 = p_n + \frac{(\delta - \beta p_{n-1})q_n}{q_{n-1}} \qquad J_n^{(0)} = -\mu q_n w_n^{(3)} \\
\rho_n^{(2)} &= \frac{\left(\rho_n^{(0)}\right)^2}{2} + q_n w_n^{(3)} \qquad J_n^{(2)} = -\mu q_n w_n^{(5)} \\
\rho_n^{(4)} &= \frac{\left(\rho_n^{(0)}\right)^3}{3} + q_n w_n^{(3)} \rho_n^{(0)} + q_n w_n^{(5)} \qquad J_n^{(4)} = -\mu q_n w_n^{(7)} \\
&\ldots \\
\rho_n^{2k} &= v_{2k} + \frac{1}{2} \sum_{l_1+l_2=2k-2} v_{l_1} v_{l_2} + \frac{1}{3} \sum_{l_1+l_2+l_3=2k-4} v_{l_1} v_{l_2} v_{l_3} + \cdots \\
&\quad + \frac{1}{k-1} \sum_{l_1+l_2+\cdots+l_{k-1}=4} v_{l_1} v_{l_2} \cdots v_{l_{k-1}} + v_0^{k-1} v_2 + \frac{v_0^{k+1}}{k+1} \\
J_n^{(2k)} &= -\mu q_n w_n^{(2k+3)} \\
&\ldots
\end{aligned} \tag{3.29}$$

and

$$\gamma_n^{(-2)} = \ln p_n \qquad z_n^{(-2)} = \mu(\delta - \beta p_{n-1})q_n/q_{n-1}$$
$$\gamma_n^{(0)} = (\beta p_n - \delta)(q_n - \beta q_{n+1})/\beta q_n \qquad z_n^{(0)} = \mu p_n(\delta - \beta p_{n-1})(q_n - \beta q_{n+1})/q_{n-1}$$
$$\gamma_n^{(2)} = (\beta p_n - \delta)\left(f_n^{(5)} - f_n^{(3)}/2\right)/q_n \qquad z_n^{(2)} = \mu(\delta - \beta p_{n-1})f_n^{(5)}/q_{n-1}$$
$$\cdots$$
$$\begin{aligned}\gamma_n^{(2k)} = g_{2k} &- \frac{1}{2}\sum_{l_1+l_2=2k-2} g_{l_1}g_{l_2} + \frac{1}{3}\sum_{l_1+l_2+l_3=2k-4} g_{l_1}g_{l_2}g_{l_3} + \cdots \\ &+ \frac{(-1)^{k-2}}{k-1}\sum_{l_1+l_2+\cdots+l_{k-1}=4} g_{l_1}g_{l_2}\ldots g_{l_{k-1}} + (-1)^{k-1}g_0^{k-1}g_2 + \frac{(-1)^k g_0^{k+1}}{k+1}\end{aligned} \tag{3.30}$$
$$z_n^{(2k)} = \mu(\delta - \beta p_{n-1})f_n^{(2k+3)}/q_{n-1}.$$
$$\cdots$$

It is interesting that infinitely many conservation laws for the special Toda-type lattice equation (1.4) can be given uniformly by equations (3.29) and (3.30) with the proper choice of parameters.

Example 2. The relativistic Volterra lattice (1.5) possesses infinitely many conservation laws (3.12) and (3.24), where the first and the second conserved densities and associated flux are written in the form,

$$\begin{aligned}\rho_n^{(0)} &= p_n + q_{n-1} + hp_{n-1}q_{n-1} \\ \rho_n^{(2)} &= \tfrac{1}{2}(p_n + q_{n-1} + hp_{n-1}q_{n-1})^2 + (1 + hp_{n-1})p_{n-1}q_{n-1} + h(1 + hp_{n-2})p_{n-1}q_{n-1}q_{n-2} \\ J_n^{(0)} &= -(1 + hp_{n-1} + hq_{n-2} + h^2p_{n-2}q_{n-2})p_{n-1}q_{n-1} \\ J_n^{(2)} &= -(1 + h(p_{n-1} + q_{n-2} + hp_{n-2}q_{n-2}))\left(p_{n-1}^2q_{n-1} + (1 + hp_{n-2})p_{n-1}q_{n-1}q_{n-2}\right) \\ &\quad - hp_{n-1}p_{n-2}q_{n-1}q_{n-2}(1 + h(p_{n-2} + q_{n-3} + hp_{n-3}q_{n-3}))\end{aligned} \tag{3.31}$$

$$\cdots$$
$$\begin{aligned}\gamma_n^{(-2)} &= \ln p_n \qquad z_n^{(-2)} = (1 + hp_{n-1})q_{n-1} \\ \gamma_n^{(0)} &= p_n + q_n + hp_nq_n \qquad z_n^{(0)} = p_nq_{n-1}(1 + hp_{n-1})(1 + hq_n).\end{aligned} \tag{3.32}$$
$$\cdots$$

The infinitely many conservation laws of the Volterra lattice equation can be derived from the above equations as $h \to 0$.

Example 3. For lattice equation (1.6), we have

$$w_n^{(1)} = \frac{\theta(n-1)(1 + p_{n-1})}{q_{n-1}\sqrt{p_{n-1}}} \qquad H_n = a_n + \lambda^{-1}b_n\Gamma_n \tag{3.33}$$

where

$$a_n = \frac{p_n + 1}{\Delta_n} \qquad b_n = \frac{q_n}{\Delta_n} \qquad \Delta_n = 1 + p_n + \frac{\theta(n-1)q_n}{q_{n-1}\sqrt{p_{n-1}}} \tag{3.34}$$

and

$$(1 + \lambda^2)\frac{\partial}{\partial t}\sum_{k=1}^{\infty}\frac{\Phi_n^k}{k} = H_n - H_{n+1} \tag{3.35}$$

where

$$\Phi_n = \sum_{k=0}^{\infty} \lambda^{2k} v_{2k}. \tag{3.36}$$

We also have

$$f_n^{(1)} = q_n \qquad G_n = \lambda c_n \Gamma_n + \tfrac{1}{2}\left(d_n + \lambda^2 e_n\right) \tag{3.37}$$

where

$$c_n = \frac{-\theta(n-1)(p_n+1)}{q_{n-1}\sqrt{p_{n-1}}\Delta_n} \qquad e_n = -d_n = \frac{\theta(n-1)q_n}{q_{n-1}\sqrt{p_{n-1}}\Delta_n} \tag{3.38}$$

and

$$(1+\lambda^2)\frac{\partial}{\partial t}\left[-\ln(p_n s_n) + \sum_{k=1}^{\infty}\frac{\Psi_n^k}{k}\right] = G_n - G_{n+1} \tag{3.39}$$

where

$$\Psi_n = \frac{r_n(\Gamma_n - \lambda q_n)}{\lambda\theta(n)\sqrt{p_n}} = \lambda^2 \sum_{k=0}^{\infty} \lambda^{2k} g_{2k} \tag{3.40}$$

with

$$g_{2k} = \frac{r_n f_n^{(2k+3)}}{\theta(n)\sqrt{p_n}}. \tag{3.41}$$

So lattice equation (1.6) admits the infinitely many conservation laws (3.12) and (3.24), where

$$\begin{aligned}
&\rho_n^{(0)} = p_n + \frac{\theta(n-1)(1+p_{n-1})q_n}{q_{n-1}\sqrt{p_{n-1}}} \qquad J_n^{(0)} = \frac{\theta(n-1)q_n\sqrt{p_{n-1}}}{q_{n-1}\Delta_n}\\
&\rho_n^{(2)} = \tfrac{1}{2}\left(\rho_n^{(0)}\right)^2 + \rho_n^{(0)} + q_n w_n^{(3)} \qquad J_n^{(2)} = b_n w_n^{(3)}\\
&\rho_n^{(4)} = \tfrac{1}{3}\left(\rho_n^{(0)}\right)^3 + \tfrac{1}{2}\left(\rho_n^{(0)}\right)^2 + \left(\rho_n^{(0)}+1\right) q_n w_n^{(3)} + q_n w_n^{(5)} \qquad J_n^{(4)} = b_n w_n^{(5)}\\
&\cdots\\
&\rho_n^{(2k)} = v_{2k} + \frac{1}{2}\sum_{l_1+l_2=2k-2} v_{l_1}v_{l_2} + \frac{1}{3}\sum_{l_1+l_2+l_3=2k-4} v_{l_1}v_{l_2}v_{l_3} + \cdots\\
&\qquad + \frac{1}{k-1}\sum_{l_1+l_2+\cdots+l_{k-1}=4} v_{l_1}v_{l_2}\cdots v_{l_{k-1}} + v_0^{k-1}v_2 + \frac{v_0^{k+1}}{k+1}\\
&\qquad + v_{2(k-1)} + \frac{1}{2}\sum_{l_1+l_2=2k-4} v_{l_1}v_{l_2} + \frac{1}{3}\sum_{l_1+l_2+l_3=2k-6} v_{l_1}v_{l_2}v_{l_3} + \cdots\\
&\qquad + \frac{1}{k-2}\sum_{l_1+l_2+\cdots+l_{k-2}=4} v_{l_1}v_{l_2}\cdots v_{l_{k-2}} + v_0^{k-2}v_2 + \frac{v_0^{k}}{k}\\
&J_n^{(2k)} = b_n w_n^{(2k+1)}\\
&\cdots
\end{aligned} \tag{3.42}$$

$$\gamma_n^{(-2)} = -\ln(\theta(n)\sqrt{p_n}) \qquad z_n^{(-2)} = \tfrac{1}{2}d_n$$

$$\gamma_n^{(0)} = -\ln(\theta(n)\sqrt{p_n}) - \frac{(1+p_n)(q_n+\theta(n)q_{n+1}/\sqrt{p_n})}{q_n} \qquad z_n^{(0)} = c_n q_n + \frac{e_n}{2}$$

$$\gamma_n^{(2)} = \frac{(1+p_n)^2(q_n+\theta(n)q_{n+1}/\sqrt{p_n})^2}{2q_n^2} - \frac{(1+p_n)(q_n+\theta(n)q_{n+1}/\sqrt{p_n})}{q_n}$$

$$z_n^{(2)} = c_n f_n^{(3)} = c_n p_n (q_n + \theta(n) q_{n+1}\sqrt{p_n})$$

$$\cdots$$

$$\begin{aligned}\gamma_n^{(2k)} = {} & g_{2k} + \frac{1}{2}\sum_{l_1+l_2=2k-2} g_{l_1}g_{l_2} + \frac{1}{3}\sum_{l_1+l_2+l_3=2k-4} g_{l_1}g_{l_2}g_{l_3} + \cdots \\ & + \frac{1}{k-1}\sum_{l_1+l_2+\cdots+l_{k-1}=4} g_{l_1}g_{l_2}\cdots g_{l_{k-1}} + g_0^{k-1}g_2 + \frac{g_0^{k+1}}{k+1} \\ & + g_{2(k-1)} + \frac{1}{2}\sum_{l_1+l_2=2k-4} g_{l_1}g_{l_2} + \frac{1}{3}\sum_{l_1+l_2+l_3=2k-6} g_{l_1}g_{l_2}g_{l_3} + \cdots \\ & + \frac{1}{k-2}\sum_{l_1+l_2+\cdots+l_{k-2}=4} g_{l_1}g_{l_2}\cdots g_{l_{k-2}} + g_0^{k-2}g_2 + \frac{g_0^{k}}{k}\end{aligned} \tag{3.43}$$

$$z_n^{(2k)} = c_n f_n^{(2k-1)}$$

$$\cdots$$

4. Integrable discretization of lattice equation (1.6)

In this section, we give an answer to the integrable discretization of lattice equation (1.6). We first derive the discrete-time approximation to equation (1.7) from the following discrete zero curvature equation:

$$\tilde{U}_n V_n = V_{n+1} U_n. \tag{4.1}$$

How do we choose a proper matrix V_n? This is a difficult problem. However, it is useful to note that

$$\frac{\tilde{\psi}_n - \psi_n}{h} = \frac{(V_n - I)\psi_n}{h}$$

and

$$\lim_{h\to 0}\frac{V_n - I}{h} = N_n \tag{4.2}$$

where I is the unit matrix. Now let us consider problem (1.9) with

$$U_n = \begin{pmatrix} \lambda p_n - \lambda^{-1} & q_n \\ r_n & \lambda s_n \end{pmatrix} \qquad V_n = I + \frac{h}{1+a\lambda^2}\begin{pmatrix} \lambda^2 f_n & \lambda u_n \\ \lambda v_n & \frac{w_n+\lambda^2 g_n}{2} \end{pmatrix} \tag{4.3}$$

where $q_n r_n = -s_n(1+p_n)$, f_n, u_n, v_n, w_n, g_n and a are determined. A direct calculation shows that

$$\tilde{U}_n V_n - V_{n+1}U_n = \frac{1+a\lambda^2}{h}\begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$\Delta_{11} = \lambda^3((f_n\tilde{p}_n - f_{n+1}p_n) + \lambda(Df_n + v_n\tilde{q}_n - u_{n+1}r_n) + \frac{\lambda(1+a\lambda^2)}{h}(\tilde{p}_n - p_n)$$

$$\Delta_{12} = \lambda^2\left(u_n\tilde{p}_n + \frac{g_n\tilde{q}_n}{2} - f_{n+1}q_n - u_{n+1}s_n\right) - u_n + \frac{w_n\tilde{q}_n}{2} + \frac{1+a\lambda^2}{h}(\tilde{q}_n - q_n)$$

$$\Delta_{21} = \lambda^2\left(f_n\tilde r_n + v_n\tilde s_n - v_{n+1}p_n - \frac{g_{n+1}r_n}{2}\right) + v_{n+1} - \frac{r_n w_{n+1}}{2} + \frac{1+a\lambda^2}{h}(\tilde r_n - r_n)$$
$$\Delta_{22} = \lambda^3\left(\frac{\tilde s_n g_n}{2} - \frac{g_{n+1}s_n}{2}\right) + \lambda\left(u_n\tilde r_n + \frac{w_n\tilde s_n}{2} - q_n v_{n+1} - \frac{s_n w_{n+1}}{2}\right) + \frac{\lambda(1+a\lambda^2)}{h}(\tilde s_n - s_n).$$

It follows that

$$\frac{\tilde p_n - p_n}{h} = f_{n+1}p_n - f_n\tilde p_n + \tilde p_n - p_n \tag{4.4}$$
$$\frac{\tilde p_n - p_n}{h} = u_{n+1}r_n - Df_n - v_n\tilde q_n \tag{4.5}$$
$$\frac{\tilde q_n - q_n}{h} = f_{n+1}q_n + u_{n+1}s_n - u_n\tilde p_n - \frac{g_n\tilde q_n}{2} + \tilde q_n - q_n \tag{4.6}$$
$$\frac{\tilde q_n - q_n}{h} = u_n - \frac{w_n\tilde q_n}{2} \tag{4.7}$$
$$\frac{\tilde r_n - r_n}{h} = v_{n+1}p_n + \frac{g_{n+1}r_n}{2} - f_n\tilde r_n - v_n\tilde s_n + \tilde r_n - r_n \tag{4.8}$$
$$\frac{\tilde r_n - r_n}{h} = \frac{r_n w_{n+1}}{2} - v_{n+1} \tag{4.9}$$
$$\frac{\tilde s_n - s_n}{h} = \frac{g_{n+1}s_n}{2} - s_n + \tilde s_n - \frac{\tilde s_n g_n}{2} \tag{4.10}$$
$$\frac{\tilde s_n - s_n}{h} = q_n v_{n+1} + \frac{s_n w_{n+1}}{2} - u_n\tilde r_n - \frac{w_n\tilde s_n}{2}. \tag{4.11}$$

Here we have chosen $a = 1 - h$. Can we obtain integrable discretization for lattice equation (1.7) from equations (4.4)–(4.11) by the proper choice of $f_n, u_n, v_n, w_n,$ and g_n? From condition (4.2) and some careful calculations, we let

$$\begin{aligned} &u_n = \frac{\tilde q_n}{\Delta_n} \qquad g_n = \frac{\tilde q_n s_{n-1}}{q_{n-1}\Delta_n} \qquad w_n = -\frac{\tilde q_n s_{n-1}}{q_{n-1}\Delta_n} \qquad f_n = \frac{1+\tilde p_n}{\Delta_n} \\ &v_n = \frac{-(1+\tilde p_n)s_{n-1}}{q_{n-1}\Delta_n} \qquad \Delta_n = 1 + \tilde p_n + \frac{\tilde q_n s_{n-1}}{q_{n-1}}. \end{aligned} \tag{4.12}$$

It follows from equations (4.4), (4.7) and (4.10) that

$$\begin{aligned} \frac{\tilde p_n - p_n}{h} &= \frac{p_n(1+\tilde p_{n+1})}{\Delta_{n+1}} - \frac{\tilde p_n(1+\tilde p_n)}{\Delta_n} + \tilde p_n - p_n \\ \frac{\tilde q_n - q_n}{h} &= \frac{\tilde q_n}{\Delta_n}\left(1 + \frac{\tilde q_n s_{n-1}}{2q_{n-1}}\right) \\ \frac{\tilde s_n - s_n}{h} &= \frac{\tilde q_{n+1}s_n^2}{2q_n\Delta_{n+1}} - \frac{\tilde q_n\tilde s_n s_{n-1}}{2q_{n-1}\Delta_n} - s_n + \tilde s_n. \end{aligned} \tag{4.13}$$

The map (4.13) is a discrete-time approximation of lattice equation (1.7). In order to prove the map is an integrable discretization, it is necessary to show that equations (4.5), (4.6), (4.8), (4.9) and (4.11) hold identically.

Note that

$$\begin{aligned} &\frac{p_n(1+\tilde p_{n+1})}{\Delta_{n+1}} - \frac{\tilde p_n(1+\tilde p_n)}{\Delta_n} + \tilde p_n - p_n + \frac{(1+p_n)\tilde q_{n+1}s_n}{q_n\Delta_{n+1}} \\ &\qquad + \frac{1+\tilde p_{n+1}}{\Delta_{n+1}} - \frac{1+\tilde p_n}{\Delta_n} - \frac{(1+\tilde p_n)\tilde q_n s_{n-1}}{q_{n-1}\Delta_n} = 0 \end{aligned} \tag{4.14}$$

$$\frac{(1+\tilde{p}_{n+1})q_n}{\Delta_{n+1}} - \frac{(1+\tilde{p}_n)\tilde{q}_n}{\Delta_n} + \frac{\tilde{q}_{n+1}s_n}{\Delta_{n+1}} - \frac{\tilde{q}_n^2 s_{n-1}}{q_{n-1}\Delta_n} + \tilde{q}_n - q_n = 0 \tag{4.15}$$

$$\frac{\tilde{q}_{n+1}s_n^2}{q_n\Delta_{n+1}} - \frac{\tilde{q}_n\tilde{s}_n s_{n-1}}{q_{n-1}\Delta_n} + \frac{(1+\tilde{p}_{n+1})s_n}{\Delta_{n+1}} - \frac{(1+\tilde{p}_n)\tilde{s}_n}{\Delta_n} + \tilde{s}_n - s_n = 0 \tag{4.16}$$

$$-\frac{(1+p_n)(1+\tilde{p}_{n+1})s_n}{q_n\Delta_{n+1}} - \frac{(1+p_n)\tilde{q}_{n+1}s_n^2}{q_n^2\Delta_{n+1}} + \frac{(1+\tilde{p}_n)^2\tilde{s}_n}{\tilde{q}_n\Delta_n} + \frac{(1+\tilde{p}_n)\tilde{s}_n s_{n-1}}{q_{n-1}\Delta_n} - \frac{(1+\tilde{p}_n)\tilde{s}_n}{\tilde{q}_n} + \frac{(1+p_n)s_n}{q_n} = 0 \tag{4.17}$$

then equations (4.5), (4.6), (4.11) and (4.8) are satisfied identically. Note that $r_n = -s_n(1+p_n)/q_n$ in the matrix U_n, we should check the consistent condition $\dot{r}_n = -\frac{\partial}{\partial t}\left[\frac{(1+p_n)s_n}{q_n}\right]$. From equation (4.9) we have

$$\dot{r}_n = \frac{s_n}{q_n(1+p_{n+1}+q_{n+1}s_n/q_n)}\left[1+p_{n+1}+\frac{(1+p_n)s_n q_{n+1}}{2q_n}\right] \tag{4.18}$$

and the consistent condition is satisfied. We thus have shown that map (4.13) is an integrable discretization of lattice equation (1.7). Here we give the proof of equations (4.14)–(4.17). Note that

$$\Delta_n = 1 + \tilde{p}_n + \frac{\tilde{q}_n s_{n-1}}{q_{n-1}}$$

we have the following equations:

$$\frac{1}{\Delta_{n+1}}\left\{p_n(1+\tilde{p}_{n+1}) + \frac{(1+p_n)\tilde{q}_{n+1}s_n}{q_n} + 1 + \tilde{p}_{n+1}\right\} = 1 + p_n \tag{4.19}$$

$$\frac{1}{\Delta_n}\left\{\tilde{p}_n(1+\tilde{p}_n) + \frac{(1+\tilde{p}_n)\tilde{q}_n s_{n-1}}{q_{n-1}} + 1 + \tilde{p}_n\right\} = 1 + \tilde{p}_n \tag{4.20}$$

$$\frac{1}{\Delta_{n+1}}\{(1+\tilde{p}_{n+1})q_n + \tilde{q}_{n+1}s_n\} = q_n \tag{4.21}$$

$$\frac{1}{\Delta_n}\left\{(1+\tilde{p}_n)\tilde{q}_n + \frac{\tilde{q}_n^2 s_{n-1}}{q_{n-1}}\right\} = \tilde{q}_n \tag{4.22}$$

$$\frac{1}{\Delta_{n+1}}\left\{(1+\tilde{p}_{n+1})s_n + \frac{\tilde{q}_{n+1}s_n^2}{q_n}\right\} = s_n \tag{4.23}$$

$$\frac{1}{\Delta_n}\left\{(1+\tilde{p}_n)\tilde{s}_n + \frac{\tilde{q}_n\tilde{s}_n s_{n-1}}{q_{n-1}}\right\} = \tilde{s}_n \tag{4.24}$$

$$\frac{(1+p_n)s_n}{q_n\Delta_{n+1}}\left\{1+\tilde{p}_{n+1}+\frac{\tilde{q}_{n+1}s_n}{q_n}\right\} = \frac{(1+p_n)s_n}{q_n} \tag{4.25}$$

$$\frac{(1+\tilde{p}_n)\tilde{s}_n}{\tilde{q}_n\Delta_n}\left\{1+\tilde{p}_n+\frac{\tilde{q}_n s_{n-1}}{q_{n-1}}\right\} = \frac{(1+\tilde{p}_n)\tilde{s}_n}{\tilde{q}_n}. \tag{4.26}$$

So equations (4.14)–(4.17) hold identically. We thus obtain an integrable discretization of lattice equation (1.6),

$$\begin{aligned} \frac{\tilde{p}_n - p_n}{h} &= \frac{p_n(1+\tilde{p}_{n+1})}{\Delta_{n+1}} - \frac{\tilde{p}_n(1+\tilde{p}_n)}{\Delta_n} + \tilde{p}_n - p_n \\ \frac{\tilde{q}_n - q_n}{h} &= \frac{\tilde{q}_n}{\Delta_n}\left(1 + \frac{\theta(n-1)\tilde{q}_n}{2q_{n-1}\sqrt{p_{n-1}}}\right) \end{aligned} \tag{4.27}$$

where

$$\Delta_n = 1 + \tilde{p}_n + \frac{\theta(n-1)\tilde{q}_n}{q_{n-1}\sqrt{p_{n-1}}}.$$

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